

Collinear triples and quadruples for Cartesian products in \mathbb{F}_p^2

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Abstract

We combine a recent point-line incidence bound of Stevens and de Zeeuw with an older lemma of Bourgain, Katz and Tao to bound the number of collinear triples and quadruples in a Cartesian product in \mathbb{F}_p^2 .

1 Introduction

Let $A \subseteq \mathbb{F}_p$ be a set in a prime order finite field of odd characteristic. In this note, which will eventually merge with [5], we combine a recent point-line incidence bound of Stevens and de Zeeuw [7] with an older lemma of Bourgain, Katz and Tao [2] to obtain an improved bound for $T(A)$, the number of collinear triples in $A \times A$, for large sets; and an optimal bound for $Q(A)$, the number of collinear quadruples in $A \times A$.

By a collinear triplet we mean an ordered triplet of three not necessarily distinct elements $(u, v, w) \in (A \times A) \times (A \times A) \times (A \times A)$ that are incident to the same line. So, for example, any point $u \in A \times A$ gives rise to the collinear triplet (u, u, u) . Collinear quadruples are defined similarly.

The quantities $T(A)$ and $Q(A)$ can be expressed in terms of the incidence function associated with $A \times A$. For a line $\ell \subset \mathbb{F}_p^2$, $i(\ell)$ equals the number of points in $A \times A$ incident to ℓ . Then

$$T(A) = \sum_{\text{all lines } \ell} i(\ell)^3 \quad \text{and} \quad Q(A) = \sum_{\text{all lines } \ell} i(\ell)^4.$$

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The contribution to collinear triples coming from the $|A|$ horizontal and the $|A|$ vertical lines incident to $|A|$ points in $A \times A$ is $2|A|^4$. All other collinear triples can be counted by the number of solutions to

$$\frac{a_1 - a_2}{a_3 - a_4} = \frac{a_1 - a_5}{a_3 - a_6} \neq 0, \quad a_i \in A, \quad a_3 - a_4, a_3 - a_6 \neq 0. \quad (1)$$

It follows from this that the expected number of collinear triples of a random set is

$$\frac{|A|^6}{p} + 2|A|^4.$$

This is because for each 5-tuple (a_1, \dots, a_5) there is a unique element $a_6 \in \mathbb{F}_p$ that satisfies (1) and it belongs to $|A|$ with probability $|A|/p$.

In view of what holds over \mathbb{R} (see the paper of Elekes and Ruzsa [3]), it is natural to expect that the bound

$$T(A) = O\left(\frac{|A|^6}{p} + \log(|A|)|A|^4\right)$$

is correct up to perhaps logarithmic factors. As we saw, sufficiently large random sets match the $|A|^6/p$ term. Sufficiently small arithmetic progressions nearly match the $\log(|A|)|A|^4$ term.

Far less is currently known. As is explained in Section 3, it is straightforward to obtain

$$\left|T(A) - \left(\frac{|A|^6}{p} + 2|A|^4\right)\right| \leq p|A|^2.$$

It follows that if $|A| = \Omega(p^{2/3})$, then $T(A) = \Theta(|A|^6/p)$.

In the range $|A| = O(p^{2/3})$, Aksoy Yazici, Murphy, Rudnev and Shkredov [1, Proposition 5], building on Rudnev's breakthrough result in [6], established the bound

$$T(A) = O(|A|^{9/2}).$$

While very strong, the result of Aksoy Yazici, Murphy, Rudnev and Shkredov, does not improve upon the range of $|A|$ where $T(A) = O(|A|^6/p)$.

Combining the two bounds above gives

$$T(A) = O\left(\frac{|A|^6}{p} + |A|^{9/2}\right).$$

Similar results are true for collinear quadruples. The expected number of collinear quadruples is $\frac{|A|^8}{p^2} + 2|A|^5$. The result of Aksoy Yazici, Murphy, Rudnev and Shkredov implies that $Q(A) = O(|A|^{11/2})$ when $|A| = O(p^{2/3})$. One expects that the correct order of magnitude up to logarithmic factors is

$$Q(A) = O\left(\frac{|A|^8}{p^2} + \log(|A|)|A|^5\right).$$

Once again large random sets and small arithmetic progressions offering (nearly) extremal examples.

We offer an improvement on the know bound for $T(A)$ when $|A| = \Omega(p^{1/2})$ and establish a nearly best possible bound for $Q(A)$.

Theorem 1. *Let $A \subseteq \mathbb{F}_p$.*

1. *The number of collinear triples in $A \times A$ satisfies*

$$T(A) = O\left(\frac{|A|^6}{p} + p^{1/2}|A|^{7/2}\right).$$

So there is at most a constant multiple of the expected number of collinear triples when $|A| = \Omega(p^{3/5})$.

2. *The number of collinear quadruples in $A \times A$ satisfies*

$$Q(A) = O\left(\frac{|A|^8}{p^2} + \log(|A|)|A|^5\right),$$

which is optimal up to perhaps logarithmic factors.

The proof of the theorem is based on a recent point-line incidence bound for Cartesian products proved by Stevens and de Zeeuw [7, Theorem 4] and a lemma of Bourgain, Katz and Tao [2, Lemma 2.1]. Applications of Theorem 1 to sum-product question in \mathbb{F}_p will be given in an expanded version of [5].

Notation. We use Landau's notation so that both statements $f = O(g)$ and $g = \Omega(f)$ mean there exists an absolute constant C such that $f \leq Cg$ and $f = \Theta(g)$ stands for $f = O(g)$ and $f = \Omega(g)$. The letter p denotes an odd prime, \mathbb{F}_p the finite field with p elements and \mathbb{F}_q^2 the 2-dimensional vector space over \mathbb{F}_p . For a line ℓ , $i(\ell)$ represents the number of points in $A \times A$ incident to ℓ .

2 The two ingredients

Let us state the two main ingredients of the proof of Theorem 1 and some straightforward consequences. We begin with the theorem of Stevens and de Zeeuw.

Theorem 2 (Stevens and de Zeeuw). *Let $A \subseteq \mathbb{F}_p$ and L be a collection of lines in \mathbb{F}_p^2 . Suppose that $|A||L| = O(p^2)$. The number of point-line incidences between $A \times A$ and L satisfies $\mathcal{I}(A \times A, L) = O(|L|^{3/4}|A|^{5/4})$.*

Stevens and de Zeeuw very reasonably imposed the additional condition $|A| < |L| < |A|^3$ because in other ranges, the Cauchy-Schwartz point-line incidence bound is better to theirs. For simplicity of argument, we omit the condition. A sanity check that allowing $|L| \leq |A|$ or $|L| \geq |A|^3$ does not hurt.

1. If $|L| \leq |A|$, we have $\mathcal{I}(A \times A, L) \leq |L||A| \leq |L|^{3/4}|A|^{5/4}$.
2. If $|A|^3 \leq |L|$, we have $\mathcal{I}(A \times A, L) \leq |L|^{1/2}|A|^2 \leq |L|^{3/4}|A|^{5/4}$.

Next we reformulate the lemma of Bourgain, Katz and Tao in terms of the incidence function i , c.f. [4, Lemma 1].

Lemma 3 (Bourgain, Katz and Tao). *Let $A \subseteq \mathbb{F}_p$.*

$$\sum_{\text{all lines } \ell} i(\ell)^2 = |A|^4 + p|A|^2.$$

In particular

$$\sum_{\text{all lines } \ell} \left(i(\ell) - \frac{|A|^2}{p} \right)^2 \leq p|A|^2.$$

The simple yet powerful lemma of Bourgain Katz, Tao was implicitly extended to general sets by Vinh [8]. The paper [4] contains other applications.

Next let M be a parameter and

$$L_M = \{M < i(\ell) < 2M\}$$

be the collection of lines from L that are incident to between M and $2M$ points in $A \times A$. We begin with an easy consequence of Lemma 3.

Lemma 4. *Let $A \subseteq \mathbb{F}_p$ and M be a real number. Suppose that $M \geq 2|A|^2/p$, then $|L_M| \leq 4p|A|^2/M^2$.*

Proof. The hypothesis $i(\ell) \geq \frac{2|A|^2}{p}$ implies that $i(\ell) - \frac{|A|^2}{p} \geq \frac{i(\ell)}{2} \geq \frac{M}{2}$.

Lemma 3 now implies

$$\frac{M^2}{4}|L_M| \leq \sum_{\ell \in L_M} \left(i(\ell) - \frac{|A|^2}{p} \right)^2 \leq \sum_{\text{all lines } \ell} \left(i(\ell) - \frac{|A|^2}{p} \right)^2 \leq p|A|^2.$$

The claim follows. \square

We now feed this bound to Theorem 2 to bound L_M . The use of Lemma 4 is a little strange, because we use it to prove that $|L_M|$ is not too big, so we may apply Theorem 2 and obtain a reasonable bound. The lemma plays a much more crucial role later.

Lemma 5. *Let $A \subseteq \mathbb{F}_p$ and $M \geq 2|A|^{3/2}/p^{1/2}$ be a real number. Then*

$$|L_M| = O\left(\frac{|A|^5}{M^4}\right).$$

In particular, under this hypothesis,

$$\sum_{\ell \in L_M} i(\ell)^3 = O\left(\frac{|A|^5}{M}\right) \text{ and } \sum_{\ell \in L_M} i(\ell)^4 = O(|A|^5).$$

Proof. The second claim follows by the first because, say,

$$\sum_{\ell \in L_M} i(\ell)^3 \leq 8M^3|L_M|.$$

To establish the first, we apply Theorem 2. Therefore we must confirm the condition $|A||L| = O(p^2)$. The hypothesis $M \geq 2|A|^{3/2}/p^{1/2}$ implies $M \geq 2|A|^2/p$. Lemma 4 can therefore be applied and in conjunction with the hypothesis $M^2 \geq 4|A|^3/p$ gives

$$|L_M| \leq \frac{4p|A|^2}{M^2} \leq \frac{p^2}{|A|}.$$

Hence, $|A||L_M| \leq p^2$ and Theorem 2 may be applied. It gives

$$M|L_M| \leq \mathcal{I}(A \times A, L_M) = O(|A|^{5/4}|L_M|^{3/4}).$$

The stated bound follows. \square

3 A straightforward bound for $T(A)$

Before proving Theorem 1, let us deduce from Lemma 3 a straightforward bound for $T(A)$.

Proposition 6. *Let $A \subseteq \mathbb{F}_p$. Then*

$$\left| T(A) - \left(\frac{|A|^6}{p} + 2|A|^4 \right) \right| \leq p|A|^3.$$

Proof. Sums are over all lines in \mathbb{F}_p^2 . We combine the first part of Lemma 3 with the identity $\sum_{\ell} i(\ell) = (p+1)|A|^2$, which follows from the fact that every point in $A \times A$ is incident to $p+1$ lines in \mathbb{F}_p^2 .

$$\begin{aligned} T(A) &= \sum_{\ell} i(\ell)^3 \\ &= \sum_{\ell} i(\ell) \left(i(\ell) - \frac{|A|^2}{p} \right)^2 + 2 \frac{|A|^2}{p} \sum_{\ell} i(\ell)^2 - \left(\frac{|A|^2}{p} \right)^2 \sum_{\ell} i(\ell) \\ &= \sum_{\ell} i(\ell) \left(i(\ell) - \frac{|A|^2}{p} \right)^2 + \frac{|A|^6}{p} + 2|A|^4 - \frac{|A|^6}{p^2}. \end{aligned}$$

The claim now follows from the fact that $i(\ell) \leq |A|$ and the second part of Lemma 3 because

$$\left| T(A) - \left(\frac{|A|^6}{p} + 2|A|^4 \right) \right| \leq \sum_{\ell} i(\ell) \left(i(\ell) - \frac{|A|^2}{p} \right)^2 \leq |A| \sum_{\ell} \left(i(\ell) - \frac{|A|^2}{p} \right)^2 \leq p|A|^3.$$

□

We therefore see that to improve the proposition, and hence the range of $|A|$ for which $T(A) = O(|A|^6/p)$, we must show that it is impossible for “the mass” of

$$\sum_{\ell} \left(i(\ell) - \frac{|A|^2}{p} \right)^2 \approx p|A|^2$$

to come from lines that satisfy $i(\ell) = \Omega(|A|)$. In other words, we must roughly speaking show that it cannot be the case that $\Omega(p)$ lines are incident to $\Omega(|A|)$ points in $A \times A$. The theorem of Stevens and de Zeeuw guarantees this. In fact Theorem 2 implies that there are $O(|A|)$ lines incident to $\Omega(|A|)$ points in $A \times A$, which is nearly optimal. To maximise the gain we perform a more careful analysis.

4 Proof of Theorem 1

For $T(A)$ we break the sum of the cubes of $i(\ell)$ in three parts:

1. Those where $i(\ell)$ is small (these give the term that resembles the expected count).
2. Those where $i(\ell)$ is of medium size (controlled by Lemma 4).
3. Those where $i(\ell)$ is large (controlled by Lemma 5 and dyadic decomposition).

The details are as follows.

$$\begin{aligned}
 T(A) &= \sum_{\text{all lines } \ell} i(\ell)^3 \\
 &= \sum_{i(\ell) \leq \frac{2|A|^2}{p}} i(\ell)^3 + \sum_{\frac{2|A|^2}{p} \leq i(\ell) \leq \frac{2|A|^{3/2}}{p^{1/2}}} i(\ell)^3 + \sum_{i(\ell) \geq \frac{2|A|^{3/2}}{p^{1/2}}} i(\ell)^3.
 \end{aligned} \tag{2}$$

The first sum is bounded using the identity $\sum_{\ell} i(\ell) = (p+1)|A|^2$:

$$\sum_{i(\ell) \leq \frac{2|A|^2}{p}} i(\ell)^3 \leq \frac{4|A|^4}{p^2} \sum_{\ell} i(\ell) = O\left(\frac{|A|^6}{p}\right).$$

The second sum is bounded using Lemma 4 and the observation that $i(\ell) \geq 2|A|^2/p$, then $i(\ell) \leq 2(i(\ell) - |A|^2/p)$:

$$\sum_{\frac{2|A|^2}{p} \leq i(\ell) \leq \frac{2|A|^{3/2}}{p^{1/2}}} i(\ell)^3 \leq \frac{2|A|^{3/2}}{p^{1/2}} \sum_{\ell} 4 \left(i(\ell) - \frac{|A|^2}{p} \right)^2 = O(p^{1/2} |A|^{7/2}).$$

The third sum is bounded using Lemma 5 and dyadic decomposition:

$$\begin{aligned}
 \sum_{i(\ell) \geq \frac{2|A|^{3/2}}{p^{1/2}}} i(\ell)^3 &= \sum_{\frac{2|A|^{3/2}}{p^{1/2}} \leq 2^j \leq \frac{|A|}{2}} \sum_{\ell \in L_{2^j}} i(\ell)^3 \\
 &= O \left(\sum_{2^j \geq \frac{2|A|^{3/2}}{p^{1/2}}} \frac{|A|^5}{2^j} \right) \\
 &= O \left(\frac{|A|^5}{\frac{|A|^{3/2}}{p^{1/2}}} \right) = O(p^{1/2} |A|^{7/2}).
 \end{aligned}$$

Substituting the three bounds into (2) gives $T(A) = O\left(\frac{|A|^6}{p} + p^{1/2}|A|^{7/2}\right)$.

A nearly identical argument works for $Q(A)$.

$$\begin{aligned} Q(A) &= \sum_{\text{all lines } \ell} i(\ell)^4 \\ &= \sum_{i(\ell) \leq \frac{2|A|^2}{p}} i(\ell)^4 + \sum_{\frac{2|A|^2}{p} \leq i(\ell) \leq \frac{2|A|^{3/2}}{p^{1/2}}} i(\ell)^4 + \sum_{i(\ell) \geq \frac{2|A|^{3/2}}{p^{1/2}}} i(\ell)^4. \end{aligned} \quad (3)$$

The first sum is bounded using the identity $\sum_{\ell} i(\ell) = (p+1)|A|^2$:

$$\sum_{i(\ell) \leq \frac{2|A|^2}{p}} i(\ell)^4 \leq \frac{8|A|^6}{p^3} \sum_{\ell} i(\ell) = O\left(\frac{|A|^8}{p^2}\right).$$

The second sum is bounded using Lemma 4:

$$\sum_{\frac{2|A|^2}{p} \leq i(\ell) \leq \frac{2|A|^{3/2}}{p^{1/2}}} i(\ell)^4 \leq \frac{4|A|^3}{p} \sum_{\ell} 4 \left(i(\ell) - \frac{|A|^2}{p} \right)^2 = O(|A|^5).$$

The third sum is bounded using Lemma 5 (and is this time of greater order of magnitude than the second):

$$\begin{aligned} \sum_{i(\ell) \geq \frac{2|A|^{3/2}}{p^{1/2}}} i(\ell)^3 &= \sum_{\frac{2|A|^{3/2}}{p^{1/2}} \leq 2^j \leq \frac{|A|}{2}} \sum_{\ell \in L_{2^j}} i(\ell)^3 \\ &= O\left(\sum_{2^j \leq |A|} |A|^5\right) \\ &= O(\log(|A|)|A|^5). \end{aligned}$$

Substituting the three bounds into (3) gives $Q(A) = O\left(\frac{|A|^8}{p^2} + \log(|A|)|A|^5\right)$.

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